Calculation by scaling of highly excited states of billiards

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We propose a method which directly gives all eigenvalues and eigenfunctions of a two-dimensional billiard in a narrow energy range by solving a generalized eigenvalue problem. This is possible through the use of scaling which allows us to write the boundary norm as a function of energy. The dimension of the problem is of the order of the number of wavelengths in the perimeter, and the number of states obtained is a fraction (0.05 or 0.1 depending on the accuracy required) of it. The method can be applied to general star-shaped domains and is illustrated for the 2×4 Bunimovich stadium where we obtain states around the 568 000th state with the precision of 10^{-4} of the mean level spacing.

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There is increasing interest in the precise and efficient calculation of highly excited states of chaotic billiards. In the context of semiclassical mechanics, the asymptotic approximation of the energy levels and the precise distribution of the level spacing is a major unsolved problem. Recently, high order corrections to the semiclassical trace formula [1] have been calculated inviting precise comparisons with exact results. The precise calculation of high single eigenfunctions is also important for the observation and understanding of scarring phenomena [2].

A method typically used consists in searching a given energy range for the zeros of some determinant [3,4]. This is a difficult task due to the complicated behavior of the determinant function. Moreover, the considered quadratic form has a numerically null subspace which corrupts the true zero eigenvalues and decreases the attainable precision. Another method consists in searching the minimum of the boundary norm [5]; this method requires not only the calculation of boundary overlaps but domain overlaps too, because the basis used is nonorthogonal over the domain (disregarding nonorthogonality, it is possible to minimize the boundary norm by the least-squares method [6]).

In this paper we propose a method which calculates directly all eigenvalues and eigenfunctions in a narrow energy range by solving a generalized eigenvalue problem in terms of quantities over the boundary. This is possible by the use of scaling which allows us to write the boundary norm explicitly as a function of the energy. The dimension of the generalized eigenvalue problem is of the order of the number of wavelengths in the perimeter.

Let $\phi(\mathbf{r})$ be a function which satisfies the Helmholtz equation $\Delta\phi(\mathbf{r})=-k_0^2\phi(\mathbf{r})$, with $k_0>0$. We associate to ϕ the scaling function $\phi(k,\mathbf{r})\equiv\phi(k\mathbf{r}/k_0)$. This family of functions depending on the scaling parameter k verifies $\Delta\phi(k,\mathbf{r})=-k^2\phi(k,\mathbf{r})$. Let $\mathcal{C}$ be a closed curve defining a star-shaped domain $\mathcal{D}$; this means that $r_n\equiv\mathbf{n}\cdot\mathbf{r}>0$, $\forall\mathbf{r}\in\mathcal{C}$ ($\mathbf{n}$ is the unit outgoing normal to $\mathcal{C}$). Then, we say that a scaling function ϕ_μ is a *scaling eigenfunction* of the billiard defined by $\mathcal{C}$, if k_μ exists such that $\phi_\mu(k_\mu,\mathbf{r})=0,\forall\mathbf{r}\in\mathcal{C}$; that is, $\phi_\mu(k_\mu,\mathbf{r})$ is an eigenfunction of the billiard with Dirichlet boundary conditions.

We define the boundary norm as $f_{[\phi]}(k)=\oint_{\mathcal{C}}\phi^2(k,\mathbf{r})dl/r_n$. The exact behavior of f as a function of k depends on ϕ; however, for normalized *scaling eigenfunctions* it satisfies a ϕ_μ independent expansion up to third order around k_μ [see (A4)]. Then, evaluating the norm and its derivative at $k_0=k_\mu+\delta_\mu$, we obtain

$$f_{[\phi_\mu]}(k_0)-\frac{\delta_\mu}{2}\frac{df_{[\phi_\mu]}}{dk}(k_0)+O(\delta_\mu^4)=0\ . \tag{1}$$

Dropping terms of $O(\delta_\mu^4)$ [7], we propose Eq. (1) as a condition leading to quantization. Using (1), we calculate all the scaling eigenfunctions with eigenvalues close to k_0 by solving the following generalized eigenvalue problem:

$$\left[\frac{dF}{dk}(k_0)-\lambda_\mu F(k_0)\right]\xi^\mu=0\ , \tag{2}$$

where F is the positive definite quadratic form associated to f. For the numerical calculation it is evaluated in some finite basis of scaling functions $\{\varphi_i(k,\mathbf{r});i=1,\ldots,N\}$,

$$F_{ij}(k_0)=\oint_{\mathcal{C}}\varphi_i(k_0,\mathbf{r})\varphi_j(k_0,\mathbf{r})\frac{dl}{r_n}\ . \tag{3}$$

The scaling eigenfunctions are found as $\phi_\mu(k,\mathbf{r})=\sum_{i=1}^N\xi_i^\mu\varphi_i(k,\mathbf{r})$, and the eigenvalues up to second order as $k_\mu=k_0-2/\lambda_\mu$. More precise eigenvalues are obtained by expanding explicitly $f_{[\phi_\mu]}$ around k_0 (up to fourth order or more) and finding its minimum. This is possible because higher order terms in (2) are almost diagonal in the basis defined by the ξ^μ's as a consequence of the quasiorthogonality of the scaling eigenfunctions over the boundary [8] (see Appendix).

The norm over the domain is $\frac{1}{4}d^2f_{[\phi_\mu]}/dk^2(k_\mu)$ [see (A2) and (A3)]. Then, with a unit domain norm the boundary norm (which is not strictly zero because we work with a finite basis) provides an estimate of the accuracy of ϕ_μ; moreover, we observe numerically that it provides a good estimate of the error in k_μ. Equation (1) is valid as long as $f_{[\phi_\mu]}(k_0)$ is large compared to $f_{[\phi_\mu]}(k_\mu)$ [to derive (A4) we have assumed $f_{[\phi_\mu]}(k_\mu)=0$]. The method becomes undetermined if k_0 coincides with an eigenvalue; then it is convenient to choose k_0 far away from an eigenvalue of the billiard. A satisfactory criterion is to require that $f_{[\phi_\mu]}(k_\mu)/f_{[\phi_\mu]}(k_0)<10^{-2}$.

We have applied this method to the 2×4 Bunimovich stadium (circle radius 1 and straight section length 2). For odd-odd states we used the following set of scaling functions: $\varphi_i(k,x,y)=\sin(xk\cos\theta_i)\sin(yk\sin\theta_i)$, with $\theta_i=(\pi/8N)(i-1/2)(5-i/N)i=1,\ldots,N$ and $N\simeq0.9k_0+20$ [9]. To solve the generalized eigenvalue problem we first diagonalized the overlap matrix F (a real symmetric matrix), and to discard the numerically null subspace we selected the eigenfunctions ψ_γ whose eigenvalues γ satisfy $\gamma/\gamma_{\max}>10^{-16}/2$ (in double precision), where $\gamma_{\max}$ is the largest eigenvalue. In the basis $\{\psi_\gamma/\sqrt{\gamma}\}$, F is the identity matrix. Then, dF/dk is diagonalized in this truncated basis. To obtain eigenvalues with higher precision, we expanded $f_{[\phi_\mu]}(k_0+\delta)$ up to 8th order in δ.

TABLE I. Eigenvalues of the 2×4 stadium computed using scaling for $k_0=1000.3848$ and $k_0=1000.4446$ ($N=920$). Moreover we give the norm over the boundary (multiplied by 10^8) of each eigenfunction. The arrows indicate in each case the position of k_0 in the spectrum.

No.	$k_0=1000.4446$ $k_\mu-1000$	$f_{[\phi_\mu]}$	$k_0=1000.3848$ $k_\mu-1000$	$f_{[\phi_\mu]}$
1	0.3526 0546	51	0.3526 0538	35
2	0.3532 8248	29	0.3532 8258	10
3	0.3543 3137	21	0.3543 3128	7
4	0.3615 1870	27	0.3615 1858	18
5	0.3623 3423	2	0.3623 3417	1
6	0.3641 6747	44	0.3641 6752	36
7	0.3678 4862	14	0.3678 4878	8
8	0.3751 7669	72	0.3751 7715	69
9	0.3762 6453	14	0.3762 6456	11
10	0.3787 2382	29	0.3787 2387	26
11	0.3807 3937	15	0.3807 3930	↙13
12	0.3908 4998	27	0.3908 4988	↖26
13	0.3937 9236	7	0.3937 9232	7
14	0.3965 2540	1	0.3965 2540	1
15	0.3987 2502	20	0.3987 2491	20
16	0.4019 9517	40	0.4019 9497	39
17	0.4050 9341	1	0.4050 9340	1
18	0.4082 4168	93	0.4082 4126	91
19	0.4111 3959	30	0.4111 3950	30
20	0.4164 8365	3	0.4164 8363	3
21	0.4179 6140	6	0.4179 6137	7
22	0.4219 2258	18	0.4219 2251	18
23	0.4288 7515	112	0.4288 7453	112
24	0.4317 2625	1	0.4317 2625	1
25	0.4344 4466	13	0.4344 4461	14
26	0.4396 1987	9	0.4396 1988	10
27	0.4419 7909	↙6	0.4419 7902	7
28	0.4478 3881	↖4	0.4478 3881	6
29	0.4512 8229	42	0.4512 8209	46
30	0.4523 4251	48	0.4523 4314	50
31	0.4587 3014	52	0.4587 3037	56
32	0.4606 5467	26	0.4606 5491	30
33	0.4674 7615	1	0.4674 7633	10
34	0.4699 5195	8	0.4699 5186	18
35	0.4716 0621	140	0.4716 0687	150
36	0.4761 0049	11	0.4761 0105	20
37	0.4769 2259	4	0.4769 2227	19

In Table I we present the calculations for $k_0=1000.3848$ and $k_0=1000.4446$ with $N=920$. They provide two independent calculations of each eigenvalue in this energy range and the difference between them gives an estimate of the precision achieved. Notice that in most cases the computed value of $f_{[\phi_\mu]}$ provides a bound to this difference. We show a range of 37 consecutive eigenvalues with an average precision of 10^{-4} of the mean level spacing. A wider range could be obtained if less precision can be tolerated, but the errors tend to increase as δ^4. The program ran on an IBM RISC/6000 work station requiring 90 min of CPU time.

The method also provides very precise eigenfunctions. To show the details that can be obtained we present in Fig. 1 a very high (around the 142 000th odd-odd state) eigenfunction of the stadium. Several aspects of its morphology are noticeable. Some "scars" are visible in the large scale picture while the detail appears quite random; this is true for most of the eigenfunctions that we have looked at although we lack a precise criterion. The structures in the eigenfunctions are more easily detected in the stellar representation (see Fig. 2) which has been recently introduced in this context [10]. In it, the Husimi distribution of the normal gradient on the boundary represents the eigenfunctions in the phase space for the Birkhoff coordinates (p=tangent unit velocity vector component versus q=arclength). The high (black) regions are the scars depicted as periodic points in the Birkhoff plane. Moreover, the eigenfunction can be completely characterized by the zeros (white) of the Husimi function and hypothesis about their distribution can be checked with greater statistical significance [11]. In general, the states listed in Table I avoid the region with p and q lower than 0.1 (see Fig. 2). On the other hand the regular "bouncing ball" type states appear concentrated in that region (e.g., $k=1000.362\,334\,17$), thus confirming Percival's conjecture [12]. Of course, as an equivalent regular region does not exist classically, this quantum effect disappears in the semiclassical limit (the white area goes to zero as $k^{-1/2}$).

In conclusion we have developed a method to compute in an efficient way the high eigenvalues and eigenfunctions of quite general two-dimensional billiards [13]. The great advantage of the method is that *all* eigenvalues and eigenfunctions in a narrow k interval are computed simultaneously with comparable accuracy, thus avoiding time consuming searches and the possibility of missing some state. The extreme accuracy achieved with this method depends critically on the presence of the denominator r_n in the boundary norm; as sketched in the Appendix, this guarantees that the *scaling eigenfunctions* are almost orthogonal. Although this condition seems to be very peculiar to billiards, we do not know at present if

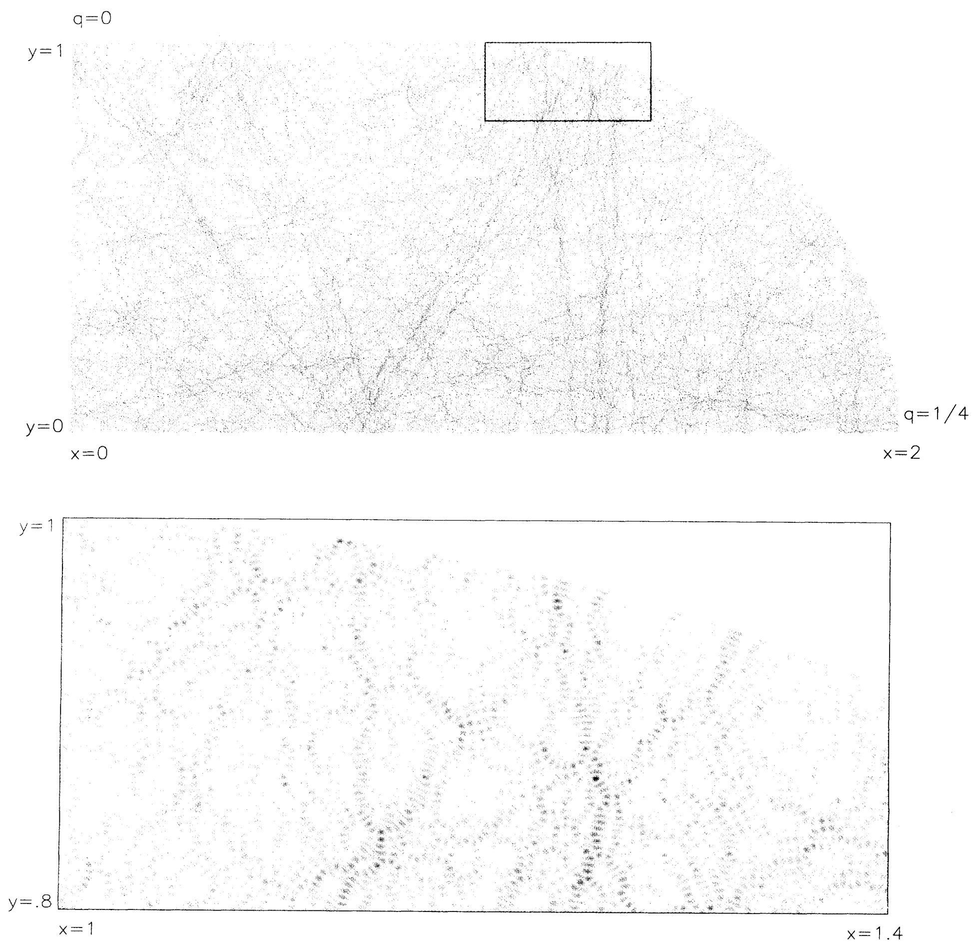

FIG. 1. (a) Linear density plot for the square of the odd-odd eigenfunction at $k = 1000.342\,508\,53$ for the 2×4 stadium. (b) Detail of the eigenfunction plotted in (a).

the method can be adapted with the same degree of accuracy to more general scaling systems.

The availability of complete stretches of eigenvalues and eigenfunctions at very high excitation is an invaluable asset for the study of the semiclassical limit of chaotic systems [14]. Calculations of spectral fluctuations, scarring phenomena, and various statistical properties become now quite feasible and are presently under way.

APPENDIX

In the following, the argument $(k_\mu, \mathbf{r}_\mathcal{C})$ is omitted, where $\mathbf{r}_\mathcal{C}$ is a point over $\mathcal{C}$, and we define $r_t \equiv \mathbf{r}\cdot\mathbf{t}$ ($\mathbf{t}$ is the unit tangent to $\mathcal{C}$). Using that $\Delta\phi_\mu = 0$ and $\partial\phi_\mu/\partial t = 0$, we expand ϕ_μ as follows:

$$\phi_\mu(k_\mu+\delta, \mathbf{r}_\mathcal{C}) = \frac{\delta}{k_\mu} r_n \frac{\partial \phi_\mu}{\partial n} + \frac{\delta^2}{k_\mu^2}\left[r_n r_t \frac{\partial^2\phi_\mu}{\partial n\,\partial t} + \frac{(r_t^2 - r_n^2)}{2}\frac{\partial^2 \phi_\mu}{\partial t^2}\right] + O(\delta^3) , \quad \text{(A1)}$$

then

$$f_{[\phi_\mu]}(k_\mu+\delta) = \frac{\delta^2}{k_\mu^2}\oint_\mathcal{C} r_n \left[\frac{\partial\phi_\mu}{\partial n}\right]^2 dl + \frac{\delta^3}{k_\mu^3}\oint_\mathcal{C}\left[r_n r_t \frac{\partial}{\partial t}\left[\frac{\partial \phi_\mu}{\partial n}\right]^2 + \frac{\partial\phi_\mu}{\partial n}(r_t^2 - r_n^2)\frac{\partial^2\phi_\mu}{\partial t^2}\right] dl + O(\delta^4) . \quad \text{(A2)}$$

Using (A2) and the following important relation [15,16]

$$\oint_\mathcal{C} r_n \left[\frac{\partial \phi_\mu}{\partial n}\right]^2 dl = 2k_\mu^2 \int_\mathcal{D} \phi_\mu^2(k_\mu, \mathbf{r})\, d\sigma , \quad \text{(A3)}$$

we obtain

$$f_{[\phi_\mu]}(k_\mu+\delta) = [2\delta^2(1-\delta/k_\mu) + O(\delta^4)]\int_\mathcal{D}\phi_\mu^2(k_\mu,\mathbf{r})\, d\sigma . \quad \text{(A4)}$$

The third order coefficient in δ was obtained integrating (A2) by parts, noting that (i) $\partial/\partial t = d/dl$, (ii)

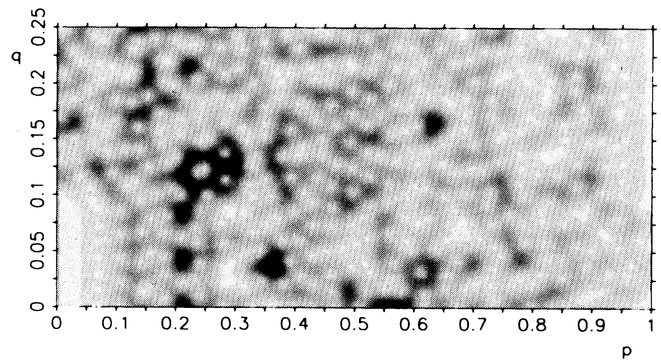

FIG. 2. Linear density plot of a reduced Husimi function in the Birkhoff coordinates (p = tangent unit velocity vector component vs arclength) for the eigenfunction plotted in Fig. 1. The dark regions are the scars depicted as periodic points in the Birkhoff plane. The white regions give (visually) the position of the zeros.

$\partial(r_n r_t)/\partial t = r_n$ and (iii) if $\mathcal{C}$ has corner points, the integration is carried about by pieces between these points, having in mind that the differential of ϕ_μ in a corner point is zero.

Scaling eigenfunctions are quasiorthogonal over the boundary. Let ϕ_μ and ϕ_ν be scaling eigenfunctions normalized over the domain with associated eigenvalues k_μ and k_ν respectively. Then, expanding these functions as in (A1),

$$\oint_{\mathcal{C}} \phi_\mu(k,\mathbf{r})\phi_\nu(k,\mathbf{r})\frac{dl}{r_n} \simeq \frac{\delta_\mu \delta_\nu}{k_\mu k_\nu} \oint_{\mathcal{C}} r_n \frac{\partial \phi_\mu}{\partial n}\frac{\partial \phi_\nu}{\partial n} dl \ . \tag{A5}$$

Working as in ([16]) to obtain (A3), it is easy to derive the following expression:

$$\oint_{\mathcal{C}} r_n \frac{\partial \phi_\mu}{\partial n}\frac{\partial \phi_\nu}{\partial n} dl = \frac{(k_\mu^2 - k_\nu^2)}{2} \int_{\mathcal{D}} (\phi_\nu \mathbf{r}\cdot\nabla\phi_\mu - \phi_\mu \mathbf{r}\cdot\nabla\phi_\nu) d\sigma \ . \tag{A6}$$

Then, using (A4), (A5), and (A6) we obtain the overlap over the boundary which results independent of k at the leading term,

$$\oint_{\mathcal{C}} \phi_\mu(k,\mathbf{r})\phi_\nu(k,\mathbf{r})\frac{dl}{r_n} \Big/ \sqrt{f_{[\phi_\mu]}(k) f_{[\phi_\nu]}(k)} \simeq \mathrm{sgn}(\delta_\mu\delta_\nu) B (k_\mu^2 - k_\nu^2)/4k_\mu k_\nu \ ,$$

where B is the domain integral in (A6). B is of order unity if we assume that the functions are not correlated.

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[8] If the scaling eigenfunctions were exactly orthogonal over the boundary (for all k), the quadratic forms $d^nF/dk^n(k_0)$ would be diagonal in the basis of scaling eigenfunctions at k_0. In such a case (2) would be an exact equation to evaluate the coefficients ξ_i^μ.
[9] This expression can be used for other billiards with perimeter $1+\pi/2$ if the calculation is carried out in double precision. A bigger symmetrized real plane wave basis does not improve the calculation because the truncated basis (see the text) dimension saturates. Higher accuracy is obtained by including symmetrized evanescent plane waves [M. V. Berry, J. Phys. A **27**, L391 (1994)]. With the inclusion of only four of these waves we have improved the precision in Table I by one order of magnitude.
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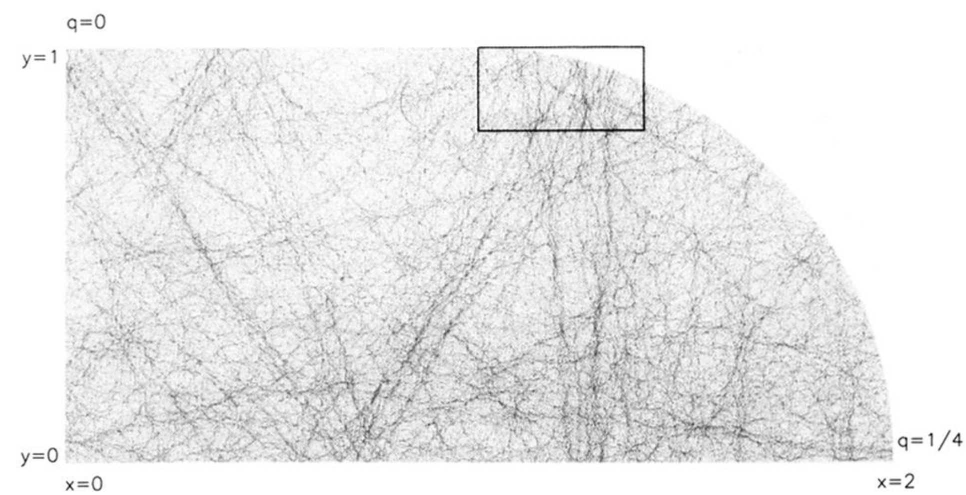

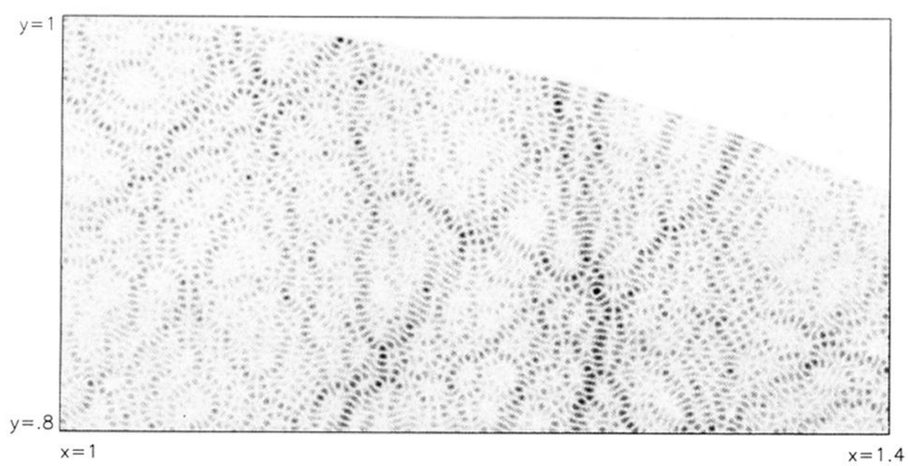

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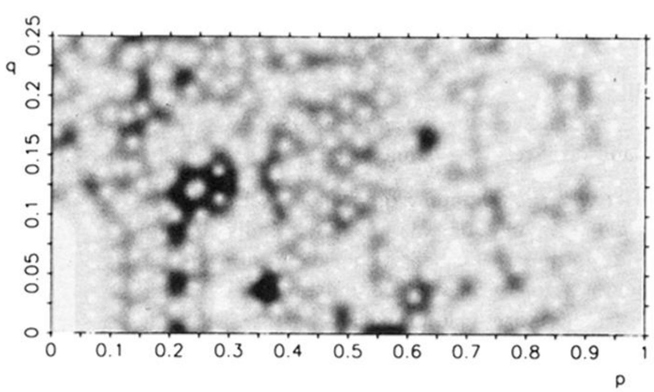

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